A Study of the Numerical Dispersion Relation for the 2-D ADI-FDTD Method

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***Abstract*—This letter presents a numerical dispersion relation for the two-dimensional (2-D) finite-difference time-domain method based on the alternating-direction implicit time-marching scheme (2-D ADI-FDTD). The proposed analytical relation for 2-D ADI-FDTD is compared with those relations in the previous works. Through numerical tests, the dispersion equation of this work was shown as correct one for 2-D ADI-FDTD.**

***Index Terms*—Dispersion relation, finite-difference time-domain (FDTD), two-dimensional (2-D) ADI-FDTD.**

I. Introduction

RECENTLY, to eliminate the Courant-Friedrich-Levy (CFL) stability constraint, the alternating-direction implicit (ADI) algorithm has been introduced to the finite-difference time-domain (FDTD) method and leads to the unconditionally stable ADI-FDTD method [1], [2]. Many researches [1]–[6] pointed out that this method has the potential to considerably reduce the number of time iterations especially in case where the fine spatial lattice relative to the wavelength is used to resolve fine geometrical features. This is mainly due to its numerical dispersion property that rapidly degrades as the simulation time step size increases. That is, in the ADI-FDTD, time step size can be determined upon not the CFL stability condition but numerical accuracy of the method such as numerical dispersion.

There are some controversies regarding the numerical dispersion relation of the two-dimensional (2-D) ADI-FDTD [1], [7] unlike the 3-D ADI-FDTD [8]. In [1], Namiki and Ito proposed an analytical dispersion relation for the 2-D ADI-FDTD and, through simulations for TE and TM wave, presented numerical phase velocities versus various time step size for the comparison with that of the traditional FDTD. However, they did not show if their closed-form relation accords with numerical results. Recently, Zhao pointed out some errors of [1] and proposed another closed-form dispersion relations for the 2-D ADI-FDTD [7]. Also, the author, according to the updating procedure, categorizes the traditional 2-D ADI-FDTD into two sub-methods and derived dispersion equations for each sub-method. Some guidelines for choosing the maximum time step size of the 2-D ADI-FDTD can be found in [7], but numerical tests to support his dispersion relations were not carried out.

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In this paper, we present a new dispersion relation for the 2-D ADI-FDTD using a similar procedure described in [8] and show that the proposed relation and numerical results are in good agreement. The unconditional stability and the relation between this work and our previous results for the dispersion equation of the 3-D ADI-FDTD method [9] are also discussed.

II. Numerical Dispersion Relation of 2-D ADI-FDTD

There are two options according to the update process in the 2-D ADI-FDTD method.

A. Option I

First, let us consider the differential form of 2-D Maxwell's equations for TM case in linear, isotropic, lossless, and nondispersive media. Following the procedure in [1] (this approach was named x-directional 2-D ADI-FDTD in [7]), the ADI time-marching algorithm approximates the 2-D differential equations in time domain as

- First iteration:

$$\varepsilon\frac{\partial E_z}{\partial t}\bigg|^{n+(1/4)}=\frac{\partial H_y}{\partial x}\bigg|^{n+(1/2)}-\frac{\partial H_x}{\partial y}\bigg|^{n} \tag{1.a}$$

$$\mu\frac{\partial H_x}{\partial t}\bigg|^{n+(1/4)}=-\frac{\partial E_z}{\partial y}\bigg|^{n} \tag{1.b}$$

$$\mu\frac{\partial H_y}{\partial t}\bigg|^{n+(1/4)}=\frac{\partial E_z}{\partial x}\bigg|^{n+(1/2)} \tag{1.c}$$

- Second iteration:

$$\varepsilon\frac{\partial E_z}{\partial t}\bigg|^{n+(3/4)}=\frac{\partial H_y}{\partial x}\bigg|^{n+(1/2)}-\frac{\partial H_x}{\partial y}\bigg|^{n+1} \tag{2.a}$$

$$\mu\frac{\partial H_x}{\partial t}\bigg|^{n+(3/4)}=-\frac{\partial E_z}{\partial y}\bigg|^{n+1} \tag{2.b}$$

$$\mu\frac{\partial H_y}{\partial t}\bigg|^{n+(3/4)}=\frac{\partial E_z}{\partial x}\bigg|^{n+(1/2)} \tag{2.c}$$

where ε, μ are the permittivity and the permeability, respectively. The superscript in (1) and (2) denotes a mapping point of the discrete time space, i.e., n means $t=n\Delta t$. After applying the central difference approximation to (1) and (2), we can get the difference update equations. The update procedure is as follows: 1) in the first iteration $E_z^{n+(1/2)}$ can be updated implicitly along the y-direction and $H_x^{n+(1/2)}$, $H_y^{n+(1/2)}$ are evaluated

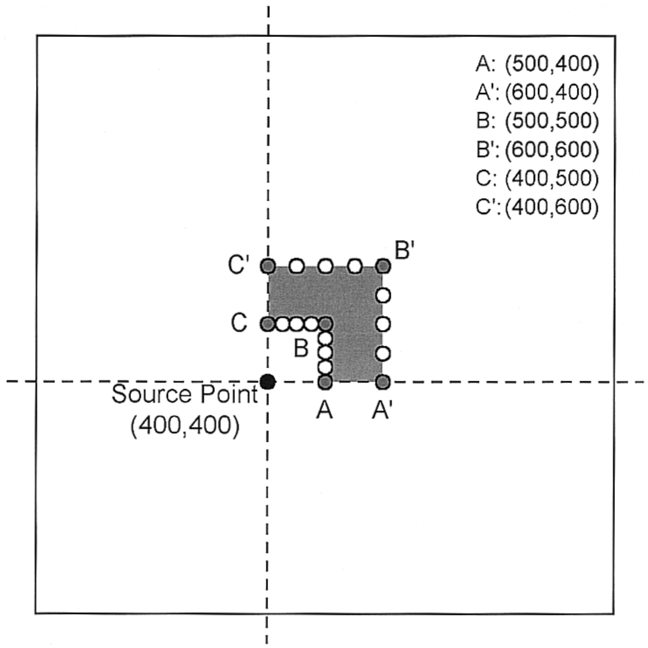

Fig. 1. Line current source model radiating in free-space for 2-D TM case. There are 18 observation points separated from the source points. The value in parenthesis is a location of the observation points. The problem size is 20 $\lambda_0 \times$ 20 λ_0.

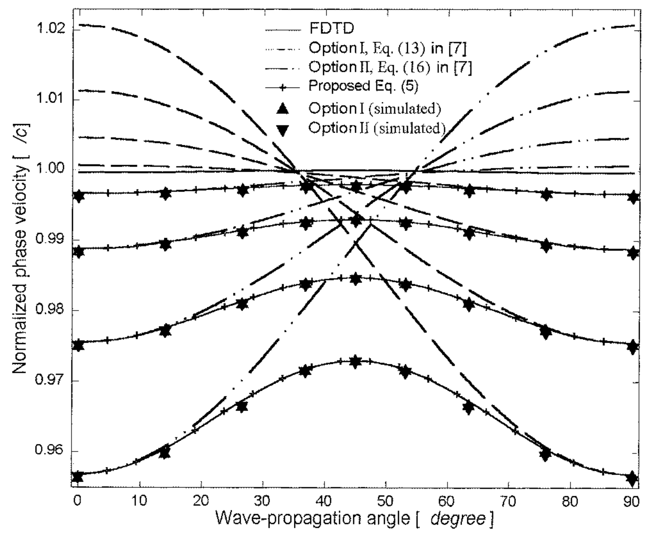

Fig. 2. Normalized numerical phase velocity of the 2-D ADI-FDTD method versus wave-propagation angle. Simulation results for two update processes and a proposed dispersion relation are in good agreement.

explicitly; 2) in the second iteration E_z^{n+1} can be updated implicitly along the x-direction and H_x^{n+1}, H_y^{n+1} are evaluated explicitly. The update procedure of [1] can be applied to the TE case also: 1) in the first iteration $E_y^{n+(1/2)}$ can be updated implicitly along the x-direction and $E_x^{n+(1/2)}$, $H_z^{n+(1/2)}$ are calculated explicitly; 2) in the second iteration E_x^{n+1} can be updated implicitly along the y-direction and E_y^{n+1}, H_z^{n+1} are calculated explicitly.

B. Option II

Option II follows the update procedure described in [2] (this approach was named y-directional 2-D ADI-FDTD in [7]). This update procedure is very similar to that of option I except that the order of the direction for implicit calculations is reversed. Therefore, we can understand that two update procedures are staggered by half time step in the time domain. For the TM case, the update process of option II is as follows: 1) in the first iteration $E_z^{n+(1/2)}$ can be updated implicitly along the x-direction and $H_x^{n+(1/2)}$, $H_y^{n+(1/2)}$ are evaluated explicitly; 2) in the second iteration E_z^{n+1} can be updated implicitly along the y-direction and H_x^{n+1}, H_y^{n+1} are evaluated explicitly. For the TE case, 1) in the first iteration $E_x^{n+(1/2)}$ can be updated implicitly along the y-direction and $E_y^{n+(1/2)}$, $H_z^{n+(1/2)}$ are calculated explicitly; 2) in the second iteration E_y^{n+1} can be updated implicitly along the x-direction and $E_x^{n+(1/2)}$, $H_z^{n+(1/2)}$ are calculated explicitly.

To derive the closed-form dispersion relation for the 2-D ADI-FDTD method, the spectral domain relationship between field components at n and $n+1$ time step can be obtained by a similar procedure described in [2] and expressed in a matrix form as

$$X^{n+1} = \Lambda X^n \tag{3}$$

where the composite vector X consists of three field components in the spectral domain, i.e., for the TM case, $X^{n+1} = [E_z^{n+1} \quad H_x^{n+1} \quad H_y^{n+1}]^T$ and $X^n = [E_z^n \quad H_x^n \quad H_y^n]^T$. Due to the limit of space, we show the matrix for only the above TM cases at the top of the next page. In the matrix Λ, $W_x = (\Delta t/\Delta x) \cdot \sin(k_x \Delta x/2)$ and $W_y = (\Delta t/\Delta y) \cdot \sin(k_y \Delta y/2)$, where k_x and k_y are spectral variables corresponding to the spatial variable x and y. We found that the Λ matrices corresponding to the above four cases (option I-TM, -TE and option II-TM, -TE) are different, but their eigenvalues are identical. The eigenvalues are 1 and one conjugate pair whose absolute values are equal to 1. This means that the 2-D ADI-FDTD method is unconditionally stable, which has been also demonstrated through numerical tests.

Now, we assume a monocromatic wave with the angular frequency ω, i.e., $X^n = X e^{j\omega n \Delta t}$. Then (3) can be expressed as

$$\left(e^{j\omega \Delta t} I - \Lambda\right) X = 0 \tag{4}$$

where I is a 3×3 identity matrix. By setting the determinant of the matrix $\left(e^{j\omega \Delta t} I - \Lambda\right)$ to be zero [8], we can derive the numerical dispersion relation for 2-D wave propagation in the ADI-FDTD method. The resulting analytical dispersion relation for the 2-D ADI-FDTD method is

$$\sin^2\left(\frac{\omega \Delta t}{2}\right) = \frac{W_x^2 W_y^2 + \mu\varepsilon W_x^2 + \mu\varepsilon W_y^2}{W_x^2 W_y^2 + \mu\varepsilon W_x^2 + \mu\varepsilon W_y^2 + \mu^2 \varepsilon^2}. \tag{5}$$

It was found that the above four update processes (option I-TM, -TE and option II-TM, -TE) result in the same 2-D dispersion equation. This result disagrees with that of [7] in that there are two different dispersion equations for option I and II in [7]. Also it should be noted that the closed form 2-D dispersion relation of (5) can be obtained straightforwardly from previous

$$\Lambda_{TM,\text{option I}} = \begin{bmatrix} \frac{\mu^2\varepsilon^2-\mu\varepsilon W_x^2-\mu\varepsilon W_y^2-W_x^2W_y^2}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{2j\mu^2\varepsilon W_y}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{-2j\mu^2\varepsilon W_x}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} \\ \frac{2j\varepsilon W_y}{\mu\varepsilon+W_y^2} & \frac{\mu\varepsilon-W_y^2}{\mu\varepsilon+W_y^2} & \frac{2W_xW_y}{\mu\varepsilon+W_y^2} \\ \frac{-2j\mu\varepsilon^2 W_x}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{2\mu\varepsilon W_xW_y}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{\mu^2\varepsilon^2-\mu\varepsilon W_x^2+\mu\varepsilon W_y^2+W_x^2W_y^2}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} \end{bmatrix}$$

$$\Lambda_{TM,\text{option II}} = \begin{bmatrix} \frac{\mu^2\varepsilon^2-\mu\varepsilon W_x^2-\mu\varepsilon W_y^2-W_x^2W_y^2}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{2j\mu^2\varepsilon W_y}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{-2j\mu^2\varepsilon W_x}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} \\ \frac{2j\mu\varepsilon^2 W_x}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{\mu^2\varepsilon^2+\mu\varepsilon W_x^2-\mu\varepsilon W_y^2+W_x^2W_y^2}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} & \frac{2\mu\varepsilon W_xW_y}{(\mu\varepsilon+W_x^2)(\mu\varepsilon+W_y^2)} \\ \frac{-2j\varepsilon W_x}{\mu\varepsilon+W_x^2} & \frac{2W_xW_y}{\mu\varepsilon+W_x^2} & \frac{\mu\varepsilon-W_x^2}{\mu\varepsilon+W_x^2} \end{bmatrix}$$

work [(3) for the 3-D dispersion relation in [9]] by assuming no change of electromagnetic properties along the z-direction, that is, by letting $W_z = 0$.

III. Numerical Verification and Discussion

To verify the numerical dispersion relation (5), we simulated a line current source radiating in free-space for two 2-D TM cases (option I and II) as shown in Fig. 1. There are 18 observation points separated from the source. We can get the normalized numerical phase velocity as described in [6]. The propagation angles of the plane wave goes through A-A', B-B', C-C' are 0°, 45°, 90°, respectively. In the simulation, to meet the purpose of the ADI-FDTD and clearly compare the results of numerical phase velocity, a fine square grid is used for all the cases with $\Delta = \Delta x = \Delta y = \lambda_0/50$. Therefore, the grid sampling density used for getting phase velocity curves is 50. In the 2-D ADI-FDTD, the time step size Δt more than the CFL stability limit Δt_{CFL} of the traditional FDTD is employed, i.e., $\mathrm{CFLN} = \Delta t/\Delta t_{CFL} = 2$, 4, 6, 8.

Fig. 2 shows normalized phase velocities of various dispersion equations and simulation results of option I-TM and option II-TM case. A numerical phase velocity for the traditional FDTD with $\mathrm{CFLN} = 1$ is also presented. Along one of the main axes (x and y), the phase velocity curves of [7] and proposed (5) are overlapped. This comes from the fact that the dispersion relation of [7] is the same as that of our work along only one of main axes. However, while a phase velocity exceeding c can be observed in [7] when the wave-propagation angle is 0° or 90°, the phase velocity of both the FDTD dispersion relation and the proposed 2-D ADI-FDTD dispersion relation (5) is always slower than light over the observation angles and has its maximum value at 45°. We can see that the phase velocity of the proposed dispersion (5) gets worse as time step size increases. When time step size is smaller than the CFL stability limit, closed-form dispersion relations show similar characteristics but data for this situation is not given. Through the numerical tests with various CFLN, it is demonstrated that the simulation results for two TM options marked at discrete angles and the phase velocity from the proposed relation are well matched. The numerical tests show that (5) is an analytical dispersion relation for the 2-D ADI-FDTD method.

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